Proceedings of the 37th IEEE
Conference on Decision & Control
Tampa, Florida USA • December 1998

LQ Optimal Design at Finitely Many Frequencies

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Abstract

The notion of Linear Quadratic (LQ) optimality at a single frequency is developed in single-input single-output (SISO) linear time-invariant (LTI) system / quasi-stationary signal framework and the optimality condition is given. LQ optimal design at finitely many frequencies is then shown to be reducible to an interpolation problem.

Keywords : Control design, LQ control.

1 Introduction

Reference tracking and disturbance rejection are two basic issues in control system design. Due to the undesirable effects of high power control inputs, perfect tracking/rejection is sacrificed by formulating the problem as the minimization of a linear quadratic cost in which the input power is also penalized. Though, optimal control based on linear quadratic methods has a well-developed theory (the reader is referred to [4] for an extensive coverage of the topics related with LQ optimal control), the case of specific reference/disturbance signals (e.g. constant, sinusoidal or periodic signals) seems to be not much elaborated. This work considers the design that achieves LQ optimality at finitely many frequencies, which is applicable to such cases.

2 LQ Optimality at a Single Frequency

The forthcoming development will consider a SISO and LTI continuous-time system in a quasi-stationary signal framework, yet the results apply to the discrete-time case as well. The well known infinite horizon (or steady-state) frequency-weighted LQ cost is defined for continuous-time systems as (see e.g. [1])

$$J_{\mathrm{LQ}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left([e(t)]^2 + [F(z)u(t)]^2\right) dt \, . \qquad (1)$$

Here e is the tracking error and u is the control input (see Figure 1), wheras F is a stable LTI filter. Minimization of J_{LQ} forces the plant output (y) to follow the reference command (r) as good as possible while keeping the power of the filtered control input small in the pass-band of F.

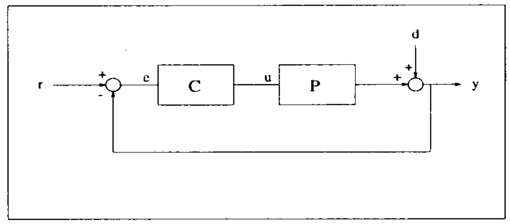

Figure 1: Unity feedback control system.

The well-known sensitivity function (S) of the control system is defined as

$$S(s) = [1 + P(s)C(s)]^{-1} \, . \qquad (2)$$

If the control system is stable and the signal $r - d$ is a quasi-stationary signal (for more information on quasi-stationarity, the reader is referred to [5]) with spectrum Φ, the LQ cost can be evaluated in the frequency domain as

$$J_{\mathrm{LQ}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} j(\omega)\Phi(\omega) d\omega \, , \qquad (3)$$

with

$$j(\omega) = \left[1 - |G(\omega)|^2\right]^{-1} \left|S(\omega) - |G(\omega)|^2\right|^2 + |G(\omega)|^2 \, , \qquad (4)$$

where $|G(\omega)|^2$ is given by

$$|G(w)|^2 = \frac{|F(\omega)|^2}{|F(\omega)|^2 + |P(\omega)|^2} \, . \qquad (5)$$

This formula motivates the following LQ optimality definition at a single frequency.

Definition 1 *The control system of Figure 1 is said to be* **LQ optimal at frequency ω**, *if $j(\omega)$ is minimum.*

The following theorem presents the simple condition for LQ optimality at a single frequency, leaving aside the degenerate cases of poles/zeroes at $\pm j\omega$.

Theorem 1 *The control system of Figure 1 is* **LQ optimal at frequency ω**, *if and only if*

$$S(\omega) = |G(\omega)|^2 \, . \qquad (6)$$

The minimum value of $j(\omega)$ is simply given by $|G(\omega)|^2$.

3 LQ Optimal Design at Finitely Many Frequencies

It is obvious from the frequency domain evaluation of the LQ cost that the case of finitely many frequencies can be treated by considering each frequency separately. The problem can then be reformulated as the determination of a stable S which satisfies $S(\omega) = |G(\omega)|^2$ at the specified frequencies. Denoting these frequencies by ω_i, $i = 1, ..., n_\omega$ and assuming that the plant has -for the sake of simplicity- simple right half-plane poles (including ∞) p_i^+, $i = 1, ..., n_{p^+}$ and zeros z_i^+, $i = 1, ..., n_{z^+}$; we can cast the LQ optimal design problem at finitely many frequencies as follows. Find a stable transfer function S which satisfies

1. $S(s)|_{s=\pm j\omega_i} = |G(\omega_i)|^2$, $i = 1, ..., n_\omega$,
2. $S(s)|_{s=p_i^+} = 0$, $i = 1, ..., n_{p^+}$,
3. $S(s)|_{s=p_i^+} = 1$, $i = 1, ..., n_{z^+}$.

The second and third conditions are necessary to prevent unstable pole/zero cancellations (see [2]). The problem can be reduced to a polynomial interpolation if a Hurwitz polynomial of degree $2n_\omega + n_{p^+} + n_{z^+}$ (e.g. $(s+\alpha)^{2n_\omega + n_{p^+} + n_{z^+}}$) is chosen as the denominator of S. The numarator polynomial can then be obtained by any interpolation algorithm (e.g. Lagrange interpolation). The feedback controller is then obtained using (2). We illustrate the procedure by a simple example.

Example: We consider the plant $P(s) = 1/(s - 0.5)$ and the frequencies $\omega = 1$ and $\sqrt{2}$ with $F = 0.3/(s + 0.5)$. As ∞ is a right half-plane zero we can set $S = 1 + N(s)/D(s)$, where D is a fifth order polynomial and N is a fourth order polynomial. Setting $D(s) = (s+1)^5$ and imposing the remaining interpolation constraints (i.e constraints at $s = 0.5$, $\pm j, \pm\sqrt{2}j$), we can obtain N by using Lagrange interpolation. This way we can result in the feedback controller

$$C(s) = \frac{5.27s^4 + 6.42s^3 + 11.22s^2 + 2.75s + 2.28}{s^4 + 0.23s^3 + 3.69s^2 + 0.63s + 2.56}.$$

We also present a disturbance rejection simulation in which a superposition of two sinusoids of frequency 1 and 2 and a white noise of comparably low variance is used as the disturbance acting on the plant. The simulation result shows that while rejecting the harmonic disturbance the white disturbance can be amplified. This is, kind of, the so-called water-bed effect (see [3]) and shows the need for the development of a design strategy which will also supply control on the overall variation of the sensitivity function.

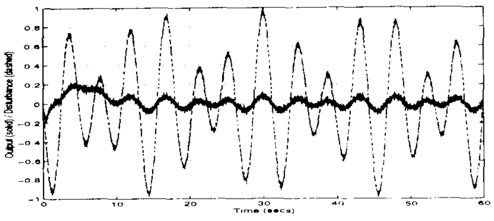

Figure 2: Example simulation (y (solid) and d (dashed)).

4 Concluding Remarks

We developed the LQ optimality notion at a single frequency and derived the condition for optimality. LQ optimal design at finitely many frequencies is then shown to be reducible to an interpolation problem. It is important to note that this framework comes into picture in the development of $\mathcal{H}_\infty$ robust control design strategies as well (refer to [2]). This means that the basic problems of robust control can be considered together with LQ optimal design at finitely many frequencies. A typical problem, which is also motivated by the simulation of the previous section, is the minimization of the $\mathcal{H}_\infty$ norm of the weighted sensitivity function (i.e. $||WS||_\infty$ where W is a proper, stable and minimum phase transfer function) together with LQ optimal design at specified frequencies. The solution of this problem will basically necessitate the use of Nevanlinna-Pick interpolation theory (see [3]). This way the overall sensitivity can be kept below a desired level while satisfying LQ optimality at the desired frequencies.

References

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